Fubini's Theorem

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Measurable and σ-Finite Measure Spaces

Measurable Space

A **measurable space** is a pair $(X, \mathcal{A})$, where:

- X is a set,
- $\mathcal{A}$ is a σ-algebra of subsets of X, i.e., a collection of subsets such that:
  1. $X \in \mathcal{A}$,
  2. If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$,
  3. If $A_1, A_2, A_3, \ldots \in \mathcal{A}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Measure Space and σ-Finiteness

A **measure space** is a triple $(X, \mathcal{A}, \mu)$ where:

- $(X, \mathcal{A})$ is a measurable space,
- $\mu : \mathcal{A} \to [0, \infty]$ is a measure satisfying:
  1. $\mu(\emptyset) = 0$,
  2. Countable additivity: for any disjoint sequence $\{A_n\} \subset \mathcal{A}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The measure space $(X, \mathcal{A}, \mu)$ is called **σ-finite** if there exists a sequence $\{X_n\}_{n=1}^{\infty} \subset \mathcal{A}$ such that:

- $X = \bigcup_{n=1}^{\infty} X_n$,
- $\mu(X_n) < \infty$ for all n.

Fubini's Theorem.

Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let $f : X \times Y \to \mathbb{R}$ be a $\mu \times \nu$-integrable function, i.e.,

$$\int_{X \times Y} |f(x, y)| \, d(\mu \times \nu)(x, y) < \infty.$$

Then:

1. $f(x, \cdot) \in L^1(Y)$ for μ-almost every $x \in X$,
2. $f(\cdot, y) \in L^1(X)$ for ν-almost every $y \in Y$,

3. The functions $x \mapsto \int_Y f(x,y)\,d\nu(y)$ and $y \mapsto \int_X f(x,y)\,d\mu(x)$ are integrable,

and
$$\int_{X\times Y} f(x,y)\,d(\mu\times\nu)(x,y) = \int_X \left(\int_Y f(x,y)\,d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f(x,y)\,d\mu(x)\right) d\nu(y).$$

Proof of Fubini's Theorem (Sketch)

Step 1: Characteristic functions. Let $f(x,y) = \chi_A(x,y)$. Then:

$$\int_{X\times Y} \chi_A\,d(\mu\times\nu) = (\mu\times\nu)(A) = \int_X \left(\int_Y \chi_A(x,y)\,d\nu(y)\right) d\mu(x).$$

Step 2: Simple functions. For a non-negative simple function

$$f(x,y) = \sum_{i=1}^{n} a_i \chi_{A_i}(x,y),$$

linearity of integration gives the desired equality.

Step 3: Non-negative measurable functions. Let $f \geq 0$. Then there exist simple $f_n \nearrow f$. By the Monotone Convergence Theorem:

$$\int_{X\times Y} f\,d(\mu\times\nu) = \int_X \left(\int_Y f(x,y)\,d\nu(y)\right) d\mu(x).$$

Step 4: Integrable functions. For general $f \in L^1(X\times Y)$, write

$$f = f^+ - f^-, \quad \text{with } f^+, f^- \in L^1.$$

Apply the above to both parts:

$$\int_{X\times Y} f = \int_X \left(\int_Y f(x,y)\,d\nu(y)\right) d\mu(x).$$

Fubini's Theorem: Detailed Proof

Fubini's Theorem

Theorem (Fubini). Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces. Let $f : X \times Y \to \mathbb{R}$ be measurable and integrable with respect to the product measure $\mu\times\nu$, i.e.,

$$\int_{X\times Y} |f(x,y)|\,d(\mu\times\nu)(x,y) < \infty.$$

Then:

1. For μ-almost every $x \in X$, the function $y \mapsto f(x,y)$ is in $L^1(Y)$,
2. For ν-almost every $y \in Y$, the function $x \mapsto f(x,y)$ is in $L^1(X)$,
3. The functions $x \mapsto \int_Y f(x,y)\,d\nu(y)$ and $y \mapsto \int_X f(x,y)\,d\mu(x)$ are integrable,
4. The following equality holds:

$$\int_{X\times Y} f(x,y)\,d(\mu\times\nu)(x,y) = \int_X \left(\int_Y f(x,y)\,d\nu(y)\right) d\mu(x) = \int_Y \left(\int_X f(x,y)\,d\mu(x)\right) d\nu(y).$$

Proof

Step 1: Characteristic functions

Let $f = \chi_A$, where $A \subset X \times Y$ is measurable. Then the function f is non-negative and simple. By Tonelli's theorem for non-negative measurable functions:

$$\int_{X\times Y} \chi_A \, d(\mu \times \nu) = \int_X \left(\int_Y \chi_A(x,y)\, d\nu(y) \right) d\mu(x) = \int_Y \left(\int_X \chi_A(x,y)\, d\mu(x) \right) d\nu(y).$$

Thus, Fubini's theorem holds for characteristic functions.

Step 2: Simple functions

Let $f(x,y) = \sum_{i=1}^n a_i \chi_{A_i}(x,y)$ be a simple, non-negative function. Since each A_i is measurable, and integrals are linear, we have:

$$\int_{X\times Y} f \, d(\mu \times \nu) = \sum_{i=1}^n a_i \int_{X\times Y} \chi_{A_i} \, d(\mu \times \nu) = \sum_{i=1}^n a_i \int_X \left(\int_Y \chi_{A_i}(x,y)\, d\nu(y) \right) d\mu(x).$$

So,

$$\int_{X\times Y} f(x,y)\, d(\mu \times \nu)(x,y) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x).$$

Similarly, the order of integration can be reversed.

Step 3: Non-negative measurable functions

Let $f : X \times Y \to [0,\infty]$ be measurable. Then there exists a sequence of simple functions $\{f_n\}$ such that $f_n \nearrow f$ pointwise. By the Monotone Convergence Theorem:

$$\int_{X\times Y} f \, d(\mu \times \nu) = \lim_{n\to\infty} \int_{X\times Y} f_n \, d(\mu \times \nu).$$

From Step 2, we know:

$$\int_{X\times Y} f_n \, d(\mu \times \nu) = \int_X \left(\int_Y f_n(x,y)\, d\nu(y) \right) d\mu(x).$$

Apply the MCT again:

$$\lim_{n\to\infty} \int_X \left(\int_Y f_n(x,y)\, d\nu(y) \right) d\mu(x) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x).$$

Hence,

$$\int_{X\times Y} f(x,y)\, d(\mu \times \nu) = \int_X \left(\int_Y f(x,y)\, d\nu(y) \right) d\mu(x).$$

The same applies with the roles of μ and ν reversed.

Step 4: General integrable functions

Let $f \in L^1(X \times Y)$. Define:

$$f^+ = \max(f, 0), \quad f^- = \max(-f, 0),$$

so that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Then both f^+ and f^- are non-negative and integrable. From Step 3, Fubini's theorem applies to each:

$$\int_{X \times Y} f^+ \, d(\mu \times \nu) = \int_X \left(\int_Y f^+(x, y) \, d\nu(y) \right) d\mu(x),$$
$$\int_{X \times Y} f^- \, d(\mu \times \nu) = \int_X \left(\int_Y f^-(x, y) \, d\nu(y) \right) d\mu(x).$$

Therefore,

$$\int_{X \times Y} f \, d(\mu \times \nu) = \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x).$$

This completes the proof. ■

Fubini's Theorem for Functions of a Complex Variable

Setup

Let $z = x + iy \in \mathbb{C}$, and define $f(x, y) = g(z) = g(x + iy)$. We wish to apply Fubini's Theorem to the function $f : \mathbb{R}^2 \to \mathbb{C}$, i.e., to interchange the order of integration:

$$\int_{\mathbb{R}^2} f(x, y) \, dx \, dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x + iy) \, dy \right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g(x + iy) \, dx \right) dy.$$

Minimal Conditions

To apply Fubini's Theorem, the function $f(x, y) = g(x + iy)$ must satisfy:

1. f is measurable on $\mathbb{R}^2$,
2. $f \in L^1(\mathbb{R}^2)$, i.e.,
$$\int_{\mathbb{R}^2} |f(x, y)| \, dx \, dy = \int_{\mathbb{R}^2} |g(x + iy)| \, dx \, dy < \infty.$$

Since $\mathbb{C} \cong \mathbb{R}^2$, the Lebesgue measure on $\mathbb{C}$ is the same as that on $\mathbb{R}^2$. Therefore, the minimal sufficient condition is:

Minimal Condition: The function $g : \mathbb{C} \to \mathbb{C}$ is measurable and Lebesgue integrable over $\mathbb{C}$:

$$\int_{\mathbb{C}} |g(z)| \, dz < \infty.$$

Measurability and Integrability of Complex-Valued Functions

Definition

Let $g : \mathbb{C} \to \mathbb{C}$. We write:

$$g(z) = u(x, y) + iv(x, y), \quad \text{where } z = x + iy.$$

Measurability

The function g is said to be **measurable** if both the real and imaginary parts $u(x, y)$ and $v(x, y)$ are Lebesgue measurable functions on $\mathbb{R}^2$.

Lebesgue Integrability

We say that $g \in L^1(\mathbb{C})$ if:

$$\int_{\mathbb{C}} |g(z)|\, dz = \int_{\mathbb{R}^2} \sqrt{u(x,y)^2 + v(x,y)^2}\, dx\, dy < \infty.$$

This ensures that g is integrable with respect to the Lebesgue measure on $\mathbb{C} \cong \mathbb{R}^2$.

Examples

Example 1: Integrable Function

Let:

$$g(z) = \frac{1}{(1 + |z|^2)^2}.$$

- g is continuous $\Rightarrow$ measurable.
- As $|z| \to \infty$, $|g(z)| \sim \frac{1}{|z|^4}$, which is integrable over $\mathbb{R}^2$.

Therefore, $g \in L^1(\mathbb{C})$ and Fubini's Theorem applies.

Example 2: Non-Integrable Function

Let:

$$g(z) = \frac{1}{|z|}.$$

- g is measurable.
- However, near $z = 0$, the integral

$$\int_{B(0,\varepsilon)} \frac{1}{|z|}\, dx\, dy = \infty.$$

Thus, $g \notin L^1(\mathbb{C})$, and Fubini's Theorem does not apply.
Failure of Fubini's Theorem for a Non-Integrable Function

The Function

Consider the function:

$$f(x, y) = \frac{1}{\sqrt{x^2 + y^2}}.$$

This is the modulus of the complex function $g(z) = \frac{1}{|z|}$, with $z = x + iy$. The function is defined on $\mathbb{R}^2 \setminus \{(0, 0)\}$ and is Lebesgue measurable.

Total Integral over $\mathbb{R}^2$

Convert to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dx\,dy = r\,dr\,d\theta.$$

Then:

$$\iint_{\mathbb{R}^2} \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} \frac{1}{r}\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\infty} 1\,dr\,d\theta = \infty.$$

Thus, $f \notin L^1(\mathbb{R}^2)$ and the full double integral diverges.

Iterated Integrals

Integrate First in y

Fix $x \in \mathbb{R}$. Then:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2+y^2}}\,dy.$$

This is an even function in y, so:

$$= 2\int_0^{\infty} \frac{1}{\sqrt{x^2+y^2}}\,dy.$$

Use the substitution $y = x\tan\theta$ (for $x \neq 0$). Then:

$$dy = x\sec^2\theta\,d\theta, \quad \sqrt{x^2+y^2} = x\sec\theta,$$

so:

$$\int_0^{\infty} \frac{1}{\sqrt{x^2+y^2}}\,dy = \int_0^{\pi/2} \frac{1}{x\sec\theta}\cdot x\sec^2\theta\,d\theta = \int_0^{\pi/2} \sec\theta\,d\theta = \infty.$$

Hence:

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2+y^2}}\,dy\right)dx = \infty.$$

Integrate First in x

The function is symmetric in x and y, so:

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^2+y^2}}\,dx\right)dy = \infty.$$

Conclusion

- The function $f(x,y) = \frac{1}{\sqrt{x^2+y^2}}$ is measurable but not integrable on $\mathbb{R}^2$.
- The total integral and both iterated integrals diverge to ∞.
- Therefore, **Fubini's Theorem does not apply**.
- In this particular case, changing the order of integration does not change the result — all expressions diverge.

Example: Fubini's Theorem Fails with Non-Absolutely Integrable Function

Function

Consider the function:

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}.$$

This can be viewed as the real part of $\frac{1}{z^2}$, where $z = x + iy$. So:

$$f(z) = \mathrm{Re}\left(\frac{1}{z^2}\right).$$

Analysis

We show that f is not absolutely integrable, yet the iterated integrals exist and differ depending on the order.

Step 1: Absolute Integrability

Convert to polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad dx\,dy = r\,dr\,d\theta.$$

Then:

$$f(x,y) = \frac{r^2\cos(2\theta)}{r^4} = \frac{\cos(2\theta)}{r^2}.$$

Then:

$$\iint_{\mathbb{R}^2} |f(x,y)|\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} \left|\frac{\cos(2\theta)}{r^2}\right| r\,dr\,d\theta = \int_0^{2\pi} |\cos(2\theta)|\,d\theta \int_0^{\infty} \frac{1}{r}\,dr = \infty.$$

So $f \notin L^1(\mathbb{R}^2)$, and Fubini's Theorem does not apply.

Step 2: Iterated Integrals

Integrate first in y:

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\right)dx.$$

Let:

$$I(x) = \int_{-\infty}^{\infty} \frac{x^2-y^2}{(x^2+y^2)^2}\,dy.$$

This integrand is **odd** in y, so the integral vanishes:

$$I(x) = 0 \quad \text{for all } x \neq 0.$$

Hence:

$$\int_{-\infty}^{\infty} I(x)\,dx = 0.$$

Integrate first in x: Now switch the order:

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\right)dy.$$

Now integrate in x. The antiderivative is known:

$$\int_{-\infty}^{\infty} \frac{x^2-y^2}{(x^2+y^2)^2}\,dx = \frac{\pi}{y}\cdot\mathrm{sgn}(y) = \pi\cdot\frac{1}{|y|}.$$

Then:

$$\int_{-\infty}^{\infty} \pi \cdot \frac{1}{|y|}\, dy = \infty.$$

So this integral diverges, but **in an asymmetric way**.
However, consider instead:

$$\int_{0}^{\infty} \left(\int_{-\infty}^{\infty} \frac{x^2 - y^2}{(x^2 + y^2)^2}\, dx \right) dy = \frac{\pi}{2}.$$

So the values depend on how the integral is taken — clear sign Fubini fails.

Conclusion

- The function $f(x, y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ is not absolutely integrable.
- The iterated integrals exist but yield different values depending on the order.
- Therefore, Fubini's Theorem does **not apply**, and interchanging the order of integration is **not valid**.